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ASYMPTOTIC BEHAVIOUR OF EISENSTEIN INTEGRALS

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Let G be a noncompact connected real semisimple Lie group with finite centre. The asymptotic behaviour of Eisenstein integrals associated with a minimal parabolic subgroup of G has to a large extent been studied by Harish-Chandra (unpublished work, see [**12**] for an account, and later in a more general setting in [**5–7**]). Other references are [**9** and **10**]. Harish-Chandra's work depends heavily on a detailed study of systems of differential equations satisfied by these integrals. In [**1**] it is shown that these systems can be transformed into complex differential equations of the regular singular type; the asymptotic behaviour of their solutions is studied by essentially applying the classical Frobenius theory.

In this announcement we present some results obtained by using another classical method, namely the representation of solutions of such equations by compact complex contour integrals (for the hypergeometric equation this method goes back to [**8**]). These integral representations can serve as the starting point for estimation by application of the method of steepest descent. This is closely connected with the use of the method of stationary phase in [**2**], where the asymptotic behaviour of Eisenstein integrals with respect to the spectral variable is studied.

I would like to thank Professor J. J. Duistermaat for suggesting this problem and for the many stimulating discussions we had.

Let $G = KAN$ be an Iwasawa decomposition. Let $\mathfrak{G}$ and $\mathfrak{A}$ be the Lie algebras of G and A, Δ the root system of $(\mathfrak{G}, \mathfrak{A})$, Δ^+ the set of positive roots corresponding to N; let $\Delta^{++} = \{\alpha \in \Delta^+; \frac{1}{2}\alpha \notin \Delta^+\}$ and let $\kappa\colon G \to K$, $H\colon G \to \mathfrak{A}$ be defined by $x \in \kappa(x) \exp H(x) N$ $(x \in G)$. Moreover, let τ_1, τ_2 be two mutually commuting representations of K in a finite dimensional complex linear space V (for convenience of notation we let them both act on the left). Let M be the centralizer of $\mathfrak{A}$ in K and set $V_M = \{v \in V; \tau_1(m)\tau_2(m)v = v\ (m \in M)\}$. For $\lambda \in \mathfrak{A}_c^*$ (the complexified dual of $\mathfrak{A}$), the

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Eisenstein integral $E(\lambda : \cdot) = E(G : P : \tau : \lambda : \cdot)\colon G \to \mathrm{End}(V)$ corresponding to the minimal parabolic subgroup $P = \mathrm{MAN}$ of G is defined by

$$E(\lambda : x) = \int_K e^{(i\lambda - \rho)H(xk)} \tau_1(\kappa(xk))\tau_2(k)\,dk \qquad (x \in G).$$

Here $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \dim(\mathfrak{G}_\alpha)\cdot\alpha$ and dk is the normalized Haar measure of K.

In this announcement we give a formula describing the asymptotic behaviour of $E(\lambda : a)|V_M$ as $a \in A$ tends to infinity along a wall of the positive Weyl chamber $A^+ = \exp(\mathfrak{A}^+)$.

Notations. If $s \in W$, the Weyl group of $(\mathfrak{G}, \mathfrak{A})$, we write $\Delta^{++}(s) = \{\alpha \in \Delta^{++}; s\alpha \in -\Delta^{++}\}$ and $\overline{N}_s = \overline{N} \cap s^{-1}Ns$, where $\overline{N} = \theta N$, θ is the Cartan involution corresponding to K. If $\lambda \in \mathfrak{A}_c^*$, $\alpha \in \Delta^{++}$, $s \in W$ we set $D_\alpha(\lambda) = \exp(4\pi(\lambda, \alpha)(\alpha, \alpha)^{-1}) - 1$ and

$$D_s(\lambda) = \prod_{\alpha\in\Delta^{++}(s)} D_\alpha(\lambda).$$

Here $(\ ,\ )$ denotes the dual of the restriction of $\mathfrak{G}_c$'s Killing form to $\mathfrak{A}_c$. If $j \in \{1, 2\}$, $s \in W$, $\lambda \in \mathfrak{A}_c^*$, $\mathrm{Im}(\lambda, \alpha) < 0$ for all $\alpha \in \Delta^{++}(s)$, then it is well known that the integral

$$I_s(\tau_j : \lambda) = \int_{\overline{N}_s} e^{-(i\lambda + \rho)H(\overline{n})} \tau_j(\kappa(\overline{n}))\,d_0\overline{n}$$

converges absolutely and defines a linear endomorphism of V_M depending holomorphically on λ. Here $d_0\overline{n}$ denotes the Haar measure of $\overline{N}_s$ corresponding to the Cartan inner product $(\cdot,\cdot) = -B(\cdot, \theta(\cdot))$.

LEMMA. *Let $s \in W$, $j \in \{1, 2\}$. Then the map*

$$\tilde{I}_s(\tau_j : \cdot)\colon \lambda \mapsto D_s(\lambda) I_s(\tau_j : \lambda)$$

extends to an entire holomorphic map $\mathfrak{A}_c^* \to \mathrm{End}(V_M)$.

INDICATION OF THE PROOF. We prove this lemma by representing $\tilde{I}_s(\tau_j : \cdot)$ as an oriented integral over a compact smooth cycle γ_s of dimension $\dim(\overline{N}_s)$ in the natural complexification $(\mathrm{Ad}_G(\overline{N}_s))_c$ of $\mathrm{Ad}_G(\overline{N}_s)$ in $(\mathrm{Aut}\ \mathfrak{G}_c)^0$. The integrand is a suitable branch of a multivalued analytic extension of $\exp[-(i\lambda + \rho)H(\cdot)]\tau_j(\kappa(\cdot))$ times an invariant holomorphic differential form and depends holomorphically on λ, whence the assertion. The cycles γ_s are first explicitly constructed for groups with $\dim(A) = 1$ and then for general groups by a multi-valued analytic continuation of the Bhanu Murti-Gindikin-Karpelevič induction procedure (cf. [3]). For the case $\tau_1 = \tau_2 =$ trivial, details can be found in [**11**].

More notations. Let S be the set of simple roots in Δ^{++}, let $F \subset S$ and let $\Delta_F = (\mathbf{Z}\cdot F) \cap \Delta$. Moreover, let $\mathfrak{A}_F = \cap_{\alpha\in F} \ker \alpha$, ${}^*\mathfrak{A} = (\mathfrak{A}_F)^\perp \cap \mathfrak{A}$, $A_F = \exp(\mathfrak{A}_F)$ and ${}^*A = \exp({}^*\mathfrak{A})$. If $C_F > 0$, ${}^*C > 0$ we put $A_F(C_F) = \{a \in A_F; a^\alpha\colon = e^{\alpha \log a} > C_F \text{ for } \alpha \in S - F\}$ and ${}^*A({}^*C) = \{{}^*a \in {}^*A; |\alpha(\log {}^*a)| < {}^*C \text{ for } \alpha \in F\}$. Let $\mathbf{C}^{S-F}$ denote the set of functions $(S - F) \to \mathbf{C}$ and let $z_F\colon A_F \to \mathbf{C}^{S-F}$ be defined by $(z_F(a))_\alpha = a^{-\alpha}$ $(\alpha \in S - F)$. If $R > 0$ we set $B_F(R) = \{z \in \mathbf{C}^{S-F}; |z_\alpha| < R\ (\alpha \in S - F)\}$. The centralizers of $\mathfrak{A}_F$ in W, K, G are denoted by W_F, K_F, M_{F1} respectively. M_{F1} is the reductive component of the standard parabolic P_F. If $\sigma \in W_F \setminus W$ (coset space), we write $w(\sigma)$ for the representative of σ in W with $\Delta^{++}(w(\sigma)^{-1}) \cap \Delta_F = \emptyset$. Then $\Delta^{++}(w(\sigma)) = \{\alpha \in \Delta^{++} - \sigma^{-1}(\Delta_F); \alpha \le 0 \text{ on } \sigma^{-1}(\mathfrak{A}_F^+)\}$. Finally we define

$$D_F(\lambda) = \prod_{\alpha \in \Delta^{++} - \Delta_F} D_\alpha(\lambda) \qquad (\lambda \in \mathfrak{A}_c^*).$$

THEOREM, PART A. *Let* $^*C > 0$. *Then there exists a constant* $C_F > 0$ *and for every* $\sigma \in W_F \setminus W$ *a map* $\Psi_{F,\sigma}\colon \mathfrak{A}_c^* \times {}^*A({}^*C) \times B_F(C_F^{-1}) \to \operatorname{End}(V_M)$, *holomorphic in the first and last and real analytic in the second variable, such that the following holds. If* $\lambda \in \mathfrak{A}_c^*$ *and* $2(\lambda, \alpha)(\alpha, \alpha)^{-1} \notin i\mathbf{Z}$ *for all* $\alpha \in \Delta^{++}$, *then*

$$E(\lambda : {}^*aa)|V_M = \sum_{\sigma \in W_F \setminus W} a^{iw(\sigma)\lambda - \rho} D_F(w(\sigma)\lambda)^{-1} \Psi_{F,\sigma}(\lambda : {}^*a : z_F(a)),$$

for every $^*a \in {}^*A({}^*C)$, $a \in A_F(C_F)$.

INDICATION OF THE PROOF. This theorem is proved by using the above lemma, Harish-Chandra's theory of the τ-radial differential equations coming from the centre of the universal enveloping algebra of $\mathfrak{G}_c$ and the techniques developed in [**11**]. In particular, an integral expression for

$$\Psi_{F,\sigma}(\lambda : {}^*a : z)$$

over the compact smooth cycle $K_F \times \gamma_{w(\sigma)} \times \gamma_{w'(\sigma)}$ is given. Here K_F denotes the centralizer of $\mathfrak{A}_F$ in K and $w'(\sigma)$ is the element of W determined by $\Delta^{++}(w'(\sigma)) = \{\alpha \in \Delta^{++} - \sigma^{-1}(\Delta_F); \alpha \ge 0 \text{ on } \sigma^{-1}(\mathfrak{A}_F^+)\}$. This integral representation will serve as the starting point for a more detailed study of $\Psi_{F,\sigma}$'s asymptotic behaviour by estimation of the integrand (cf. also [**11**]). In particular, substitution of $z = 0$ in the integral straightforwardly gives

THEOREM, PART B. *Let* $\sigma \in W_F \setminus W$. *Then for all* $\lambda \in \mathfrak{A}_c^*$, $^*a \in {}^*A$ *we have*

$$\Psi_{F,\sigma}(\lambda : {}^*a : 0) = \frac{\operatorname{vol}(K)}{\operatorname{vol}(K_F)}.$$

$$E(M_{F1} : (P \cap M_{F1}) : \tau : w(\sigma)\lambda : {}^*a) \circ [\tau_1(w(\sigma))\tau_2(w'(\sigma))] \circ \tilde{I}_{w(\sigma)}(\tau_1 : -\lambda) \circ \tilde{I}_{w'(\sigma)}(\tau_2 : \lambda).$$

Here $\operatorname{vol}(\cdot)$ *denotes the volume with respect to the Cartan inner product on* $\mathfrak{G}$.

REMARK. For real values of λ, the formulas in the above theorem agree with Harish-Chandra's theory of the constant terms of Eisenstein integrals (cf. [**4–7**]).

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